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Chains of twists for classical Lie algebras

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Abstract. For chains of regular injections $\mathcal{A}_p \subset \mathcal{A}_{p-1} \subset \cdots \subset \mathcal{A}_1 \subset \mathcal{A}_0$ of Hopf algebras the sets of maximal extended Jordanian twists $\{\mathcal{F}_{\mathcal{E}_k}\}$ are considered. We prove that under certain conditions there exists for $\mathcal{A}_0$ the twist $\mathcal{F}_{\mathcal{B}_{k<0}}$ composed of the factors $\mathcal{F}_{\mathcal{E}_k}$. The general construction of a chain of twists is applied to the universal envelopings $U(\mathfrak{g})$ of classical Lie algebras $\mathfrak{g}$. We study the chains for the infinite series A_n, B_n and D_n. The properties of the deformation produced by a chain $U(\mathfrak{g})_{\mathcal{F}_{\mathcal{B}_{k<0}}}$ are explicitly demonstrated for the case of $\mathfrak{g} = so(9)$.

1. Introduction

The triangular Hopf algebras and twists (they preserve the triangularity [1, 2]) play an important role in quantum group theory and applications [3–5]. Very few types of twists were written explicitly in a closed form. The well known example is the Jordanian twist ($\mathcal{JT}$) of the Borel algebra $B(2)$ ($\{H, E|[H, E] = E\}$) with $r = H \wedge E$ [6] where the triangular R-matrix $\mathcal{R} = (\Phi_{\mathcal{J}})_{21}\Phi_{\mathcal{J}}{}^{-1}$ is defined by the twisting element [7, 8]

$$\Phi_{\mathcal{J}} = \exp\{H \otimes \sigma\} \tag{1.1}$$

with $\sigma = \ln(1 + E)$. In [9] it was shown that there exist different extensions ($\mathcal{ET}$'s) of this twist. Using the notion of a factorizable twist [10], the element $\mathcal{F}_{\mathcal{E}} \in \mathcal{U}(sl(N))^{\otimes 2}$,

$$\mathcal{F}_{\mathcal{E}} = \Phi_{\mathcal{E}}\Phi_{\mathcal{J}} = \left(\prod_{i=2}^{N-2} \Phi_{\mathcal{E}_i}\right)\Phi_{\mathcal{J}} = \exp\left\{2\xi \sum_{i=2}^{N-1} E_{1i} \otimes E_{iN}\mathrm{e}^{-\widetilde{\sigma}}\right\} \exp\{H \otimes \widetilde{\sigma}\} \tag{1.2}$$

was proved to satisfy the twist equation, where $E = E_{1N}$, $H = E_{11} - E_{NN}$ is one of the Cartan generators $H \in \mathfrak{h}(sl(N))$, $\widetilde{\sigma} = \frac{1}{2}\ln(1 + 2\xi E)$ and $\{E_{ij}\}_{i,j=1,\ldots,N}$ is the standard $gl(N)$ basis.

Any subset of exponentials $\{\Phi_{\mathcal{E}_i} \mid i = 2, \ldots, N-2\}$ can be used to form an extended twist like (1.2). This means that similar extended twistings can be applied to different algebras with similar structure. In this particular case it is not difficult to explain this effect: the factors $\Phi_{\mathcal{E}_i}$ commute and the subalgebras where they are defined (the carrier subalgebras [8]) intersect by the central element E.

Let $\mathcal{A}$ be a Hopf algebra, $\mathcal{B}$ and $\mathcal{C}$ be its subalgebras such that they are carriers for twists $\mathcal{F}_{\mathcal{B}}$ and $\mathcal{F}_{\mathcal{C}}$, respectively. It is important to know under what conditions the sequence $\mathcal{F}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}}$ provides a new twisting element and what are its properties. In this paper we study the

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possibility to compose extended twists for the universal enveloping algebras of classical Lie algebras.

In section 2 we present a short list of basic relations for twists. The general properties of extended twists are displayed in section 3. The sufficient conditions for the existence of a composition of twists defined for subalgebras are formulated in section 4. In section 5 the same problem is solved for chains of subalgebras. It is proved that the corresponding chains of twists $\mathcal{F}_{\mathcal{B}_{k<0}}$ exist in classical Lie algebras of the series A, B and D. Using the regular injection $A_{n-1} \longrightarrow C_n$ one can implement into $U(C_n)$ a chain typical for A_{n-1}. Such improper chains are studied in section 6. The properties of twisting performed by a chain $\mathcal{F}_{\mathcal{B}_{k<0}}$ are illustrated by the explicit example of a deformation $U(so(9)) \stackrel{\mathcal{F}_{\mathcal{B}_{k<0}}}{\longrightarrow} U_{\mathcal{F}_{\mathcal{B}_{k<0}}}(so(9))$ presented in section 7. We conclude with some brief remarks about the possible multiparametrization of chains and the corresponding deformations.

2. Basic definitions

In this section we remind the reader briefly of the basic properties of twists.

A Hopf algebra $\mathcal{A}(m, \Delta, \epsilon, S)$ with multiplication $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, coproduct $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, counit $\epsilon: \mathcal{A} \to C$ and antipode $S: \mathcal{A} \to \mathcal{A}$ can be transformed [1] by an invertible (twisting) element $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$, $\mathcal{F} = \sum f_i^{(1)} \otimes f_i^{(2)}$, into a twisted one $\mathcal{A}_{\mathcal{F}}(m, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}})$. This Hopf algebra $\mathcal{A}_{\mathcal{F}}$ has the same multiplication and counit but the twisted coproduct and antipode given by

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1} \qquad S_{\mathcal{F}}(a) = vS(a)v^{-1} \tag{2.1}$$

with

$$v = \sum f_i^{(1)} S(f_i^{(2)}) \qquad a \in \mathcal{A}.$$

The twisting element has to satisfy the equations

$$(\epsilon \otimes id)(\mathcal{F}) = (id \otimes \epsilon)(\mathcal{F}) = 1 \tag{2.2}$$

$$\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F}). \tag{2.3}$$

The first one is just a normalization condition and follows from the second relation modulo a non-zero scalar factor.

If $\mathcal{A}$ is a Hopf subalgebra of $\mathcal{B}$ the twisting element $\mathcal{F}$ satisfying (2.2) and (2.3) induces the twist deformation $\mathcal{B}_{\mathcal{F}}$ of $\mathcal{B}$. In this case one can put $a \in \mathcal{B}$ in all the formulae (2.1). This will completely define the Hopf algebra $\mathcal{B}_{\mathcal{F}}$. Let $\mathcal{A}$ and $\mathcal{B}$ be the universal enveloping algebras: $\mathcal{A} = U(\mathfrak{l}) \subset \mathcal{B} = U(\mathfrak{g})$ with $\mathfrak{l} \subset \mathfrak{g}$. If $U(\mathfrak{l})$ is the minimal subalgebra on which $\mathcal{F}$ is completely defined as $\mathcal{F} \in U(\mathfrak{l}) \otimes U(\mathfrak{l})$ then $\mathfrak{l}$ is called the carrier algebra for $\mathcal{F}$ [8].

The composition of appropriate twists can be defined as $\mathcal{F} = \mathcal{F}_2\mathcal{F}_1$. Here the element $\mathcal{F}_1$ has to satisfy the twist equation with the coproduct of the original Hopf algebra, while $\mathcal{F}_2$ must be its solution for $\Delta_{\mathcal{F}_1}$ of the algebra twisted by $\mathcal{F}_1$.

If the initial Hopf algebra $\mathcal{A}$ is quasitriangular with the universal element $\mathcal{R}$ then so is the twisted one $\mathcal{A}_{\mathcal{F}}(m, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ with

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\,\mathcal{R}\,\mathcal{F}^{-1}. \tag{2.4}$$

Most of the explicitly known twisting elements have the factorization property with respect to comultiplication

$$(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}\mathcal{F}_{13} \qquad \text{or} \qquad (\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23}$$

and

$$(id \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13} \qquad \text{or} \qquad (id \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{12}.$$

To guarantee the validity of the twist equation, these identities are to be combined with the additional requirement $\mathcal{F}_{12}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{12}$ or the Yang–Baxter equation on $\mathcal{F}$ [10, 11].

An important subclass of factorizable twists consists of elements satisfying the equations

$$(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23} \tag{2.5}$$

$$(id \otimes \Delta_{\mathcal{F}})(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13}. \tag{2.6}$$

Apart from the universal R-matrix $\mathcal{R}$ that satisfies these equations for $\Delta_{\mathcal{F}} = \Delta^{op}$ ($\Delta^{op} = \tau \circ \Delta$, where $\tau(a \otimes b) = b \otimes a$) there are two more well developed cases of such twists: the Jordanian twist of the Borel algebra $B(2)$ where $\mathcal{F}_j$ has the form (1.1) (see [7]) and the extended Jordanian twists (see [9, 12, 13] for details).

According to the result by Drinfeld [2] skew (constant) solutions of the classical Yang–Baxter equation (CYBE) can be quantized and the deformed algebras thus obtained can be presented in the form of twisted universal enveloping algebras. On the other hand, such solutions of CYBE can be connected with the quasi-Frobenius carrier subalgebras of the initial classical Lie algebra [14]. A Lie algebra $\mathfrak{g}(\mu)$, with the Lie composition μ, is called Frobenius if there exists a linear functional $g^* \in \mathfrak{g}^*$ such that the form $b(g_1, g_2) = g^*(\mu(g_1, g_2))$ is non-degenerate. This means that $\mathfrak{g}$ must have a non-degenerate 2-coboundary $b(g_1, g_2) \in B^2(\mathfrak{g}, \boldsymbol{K})$. The algebra is called quasi-Frobenius if it has a non-degenerate 2-cocycle $b(g_1, g_2) \in Z^2(\mathfrak{g}, \boldsymbol{K})$ (not necessarily a coboundary). The classification of quasi-Frobenius subalgebras in $sl(n)$ was given in [14].

The deformations of quantized algebras include the deformations of their Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$. Let us fix one of the constituents $\mathfrak{g}_1^*(\mu_1^*)$ (with composition μ_1^*) and deform it in the first order,

$$(\mu_1^*)_t = \mu_1^* + t\mu_2^*$$

its deforming function μ_2^* is also a Lie product and the deformation property becomes reciprocal: μ_1^* can be considered as a first-order deforming function for the algebra $\mathfrak{g}_2^*(\mu_2^*)$. Let $\mathfrak{g}(\mu)$ be a Lie algebra that forms Lie bialgebras with both $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$. This means that we have a one-dimensional family $\{(\mathfrak{g}, (\mathfrak{g}_1^*)_t)\}$ of Lie bialgebras and correspondingly a one-dimensional family of quantum deformations $\{\mathcal{A}_t(\mathfrak{g}, (\mathfrak{g}_1^*)_t)\}$ [15]. This situation provides the possibility to construct in the set of Hopf algebras a smooth curve connecting quantizations of the type $\mathcal{A}(\mathfrak{g}, \mathfrak{g}_1^*)$ with those of $\mathcal{A}(\mathfrak{g}, \mathfrak{g}_2^*)$. Such smooth transitions can involve contractions provided $\mu_2^* \in B^2(\mathfrak{g}_1^*, \mathfrak{g}_1^*)$. This happens in the case of $\mathcal{JT}, \mathcal{ET}$ and some other twists (see [16] and references therein).

3. Extended twists

Extended Jordanian twists are associated with the set $\{\boldsymbol{L}(\alpha, \beta, \gamma, \delta)_{\alpha+\beta=\delta}\}$ of Frobenius algebras [9, 12]

$$\begin{aligned} &[H, E] = \delta E \qquad [H, A] = \alpha A \qquad [H, B] = \beta B \\ &[A, B] = \gamma E \qquad [E, A] = [E, B] = 0 \qquad \alpha + \beta = \delta. \end{aligned} \tag{3.1}$$

For limit values of γ and δ the structure of $\boldsymbol{L}$ degenerates. For the internal (non-zero) values of γ and δ the twists associated with the corresponding $\boldsymbol{L}$'s are equivalent. It is sufficient to study the one-dimensional subvariety $\mathcal{L} = \{\boldsymbol{L}(\alpha, \beta)_{\alpha+\beta=1}\}$, that is to consider the carrier algebras

$$\begin{aligned} &[H, E] = E \qquad [H, A] = \alpha A \qquad [H, B] = \beta B \\ &[A, B] = E \qquad [E, A] = [E, B] = 0 \qquad \alpha + \beta = 1. \end{aligned} \tag{3.2}$$

The corresponding group 2-cocycles (twists) are

$$\mathcal{F}_{\mathcal{E}(\alpha,\beta)} = \Phi_{\mathcal{E}(\alpha,\beta)}\Phi_j \tag{3.3}$$

or

$$\mathcal{F}_{\mathcal{E}'(\alpha,\beta)} = \Phi_{\mathcal{E}'(\alpha,\beta)}\Phi_j \tag{3.4}$$

with

$$\begin{aligned}
&\Phi_j = \exp\{H \otimes \sigma\} \\
&\Phi_{\mathcal{E}(\alpha,\beta)} = \exp\{A \otimes B\mathrm{e}^{-\beta\sigma}\} \\
&\Phi_{\mathcal{E}'(\alpha,\beta)} = \exp\{-B \otimes A\mathrm{e}^{-\alpha\sigma}\}.
\end{aligned} \tag{3.5}$$

Twists (3.3) and (3.4) define the deformed Hopf algebras $\boldsymbol{L}_{\mathcal{E}(\alpha,\beta)}$ with the costructure

$$\begin{aligned}
&\Delta_{\mathcal{E}(\alpha,\beta)}(H) = H \otimes \mathrm{e}^{-\sigma} + 1 \otimes H - A \otimes B\mathrm{e}^{-(\beta+1)\sigma} \\
&\Delta_{\mathcal{E}(\alpha,\beta)}(A) = A \otimes \mathrm{e}^{-\beta\sigma} + 1 \otimes A \\
&\Delta_{\mathcal{E}(\alpha,\beta)}(B) = B \otimes \mathrm{e}^{\beta\sigma} + \mathrm{e}^{\sigma} \otimes B \\
&\Delta_{\mathcal{E}(\alpha,\beta)}(E) = E \otimes \mathrm{e}^{\sigma} + 1 \otimes E = E \otimes 1 + 1 \otimes E + E \otimes E
\end{aligned} \tag{3.6}$$

and $\boldsymbol{L}_{\mathcal{E}'(\alpha,\beta)}$ defined by

$$\begin{aligned}
&\Delta_{\mathcal{E}'(\alpha,\beta)}(H) = H \otimes \mathrm{e}^{-\sigma} + 1 \otimes H + B \otimes A\mathrm{e}^{-(\alpha+1)\sigma} \\
&\Delta_{\mathcal{E}'(\alpha,\beta)}(A) = A \otimes \mathrm{e}^{\alpha\sigma} + \mathrm{e}^{\sigma} \otimes A \\
&\Delta_{\mathcal{E}'(\alpha,\beta)}(B) = B \otimes \mathrm{e}^{-\alpha\sigma} + 1 \otimes B \\
&\Delta_{\mathcal{E}'(\alpha,\beta)}(E) = E \otimes \mathrm{e}^{\sigma} + 1 \otimes E = E \otimes 1 + 1 \otimes E + E \otimes E.
\end{aligned} \tag{3.7}$$

The sets $\{\boldsymbol{L}_{\mathcal{E}(\alpha,\beta)}\}$ and $\{\boldsymbol{L}_{\mathcal{E}'(\alpha,\beta)}\}$ are equivalent due to the Hopf isomorphism $\boldsymbol{L}_{\mathcal{E}(\alpha,\beta)} \approx \boldsymbol{L}_{\mathcal{E}'(\beta,\alpha)}$:

$$\{\boldsymbol{L}_{\mathcal{E}}(\alpha,\beta)\} \approx \{\boldsymbol{L}_{\mathcal{E}'}(\alpha,\beta)\} \approx \{\boldsymbol{L}_{\mathcal{E}}(\alpha \geqslant \beta)\} \cup \{\boldsymbol{L}_{\mathcal{E}'}(\alpha \geqslant \beta)\}. \tag{3.8}$$

So, it is sufficient to use only one of the extensions either $\Phi_{\mathcal{E}(\alpha,\beta)}$ or $\Phi_{\mathcal{E}'(\alpha,\beta)}$, or a half of the domain for (α, β).

The set $\mathcal{L} = \{\boldsymbol{L}(\alpha,\beta)_{\alpha+\beta=1}\}$ is just the family of four-dimensional Frobenius algebras that one finds in $U(sl(N))$ [14].

4. Sequences of twists

Consider again the formula (1.2) (from now on we use a basis normalized as in (3.2), so here $H = \frac{1}{2}(E_{11} - E_{NN})$),

$$\mathcal{F}_{\mathcal{E}} = \left(\prod_{i=2}^{N-1} \Phi_{\mathcal{E}_i}\right)\Phi_{\mathcal{J}} = \left(\prod_{i=2}^{N-1} \exp\left\{E_{1i} \otimes E_{iN}\mathrm{e}^{-\frac{1}{2}\sigma}\right\}\right)\exp\{H \otimes \sigma\}. \tag{4.1}$$

In the product of exponentials each factor $\Phi_{\mathcal{E}_r}$ is itself a twisting element for the Hopf algebra previously twisted by $(\prod_{i=2}^{r-1} \Phi_{\mathcal{E}_i})\Phi_{\mathcal{J}}$. This is a very simple example of a chain of twists. All the factors $\Phi_{\mathcal{E}_i}$ commute and the corresponding twistings can be performed in an arbitrary order. Nevertheless, as we shall see this construction plays an important role in composing non-trivial chains.

The previous example also demonstrates that it is worth searching the conditions which will guarantee that each member of a sequence of elements of the type $\Phi_{\mathcal{E}_i}$ is the solution of equations (2.3) for coproducts defined by all the previous twists of the sequence.

One of the obvious solutions to this problem can be formulated as follows.

Proposition 1. *Let $\mathcal{A}$ be a Hopf algebra, $\mathcal{B}$ and $\mathcal{C}$ be its subalgebras such that they are carriers for twists $\mathcal{F}_\mathcal{B}$ and $\mathcal{F}_\mathcal{C}$, respectively. Let $\mathcal{F}_\mathcal{B}$ commute with $\Delta\mathcal{C}$. Then $\mathcal{C}$ is stable with respect to $\mathcal{F}_\mathcal{B}$, $\mathcal{F}_\mathcal{C}$ is a twisting element for $\mathcal{A}_{\mathcal{F}_\mathcal{B}}$ and the composition $\mathcal{F}_\mathcal{C}\mathcal{F}_\mathcal{B}$ is a twisting element for $\mathcal{A}$.*

In the previous example $\mathcal{B}$ and $\mathcal{C}$ were the Heisenberg subalgebras in $sl(N)$ intersecting by the element E_{1N}. The other trivial case is when $\mathcal{A}$ contains the direct sum $\mathcal{B} \oplus \mathcal{C}$. In the next section we shall study some non-trivial cases typical for the universal enveloping classical Lie algebras.

5. Chains

For the classical Lie algebras there exists the possibility of constructing sequences of carrier subalgebras systematically.

Proposition 2. *Let $\mathcal{A}$ be a Hopf algebra and*

$$\mathcal{A}_p \subset \mathcal{A}_{p-1} \subset \cdots \subset \mathcal{A}_1 \subset \mathcal{A}_0 \equiv \mathcal{A} \tag{5.1}$$

a sequence of Hopf subalgebras such that

$$\mathcal{B}_k \subset \mathcal{A}_k \qquad k = 0, 1, \ldots, p \tag{5.2}$$

are the carrier subalgebras for twisting elements $\mathcal{F}_{\mathcal{B}_k}$. Let $\mathcal{F}_{\mathcal{B}_k}$ commute with $\Delta\mathcal{A}_{k+1}$:

$$\mathcal{F}_{\mathcal{B}_k}\Delta\mathcal{A}_{k+1} = \Delta\mathcal{A}_{k+1}\mathcal{F}_{\mathcal{B}_k}. \tag{5.3}$$

Then for any $k = 0, 1, \ldots, p$ the composition

$$\mathcal{F}_{\mathcal{B}_{k<0}} \equiv \mathcal{F}_{\mathcal{B}_k}\mathcal{F}_{\mathcal{B}_{k-1}} \ldots \mathcal{F}_{\mathcal{B}_0} \tag{5.4}$$

is a twisting element for $\mathcal{A}$.

Now we shall show how this scheme can be realized for the universal enveloping algebras $U(\mathfrak{g})$ for classical Lie algebras $\mathfrak{g}$ ($U(\mathfrak{g})$ is considered here as a Hopf algebra with primitive comultiplication of generators). The construction will be similar for the classical series A_n, B_n and D_n. In the case of symplectic algebras C_n the chain would not be completely proper and we shall treat this situation separately.

Let us consider the following sequences of Hopf algebras:

$$U(sl(N)) \supset U(sl(N-2)) \supset \cdots \supset U(sl(N-2k)) \supset \cdots \qquad \text{for} \quad A_{N-1} \tag{5.5}$$

$$U(so(2N)) \supset U(so(2(N-2)) \supset \cdots \supset U(so(2(N-2k)) \supset \cdots \qquad \text{for} \quad D_N \tag{5.6}$$

$$U(so(2N+1)) \supset U(so(2(N-2)+1) \supset \cdots \supset U(so(2(N-2k)+1) \supset \cdots \qquad \text{for} \quad B_N. \tag{5.7}$$

We want to show that for these sequences there exist the sets of maximal $\mathcal{ET}$'s with the properties listed in proposition 2. In each element $\mathcal{A}_k$ of the sequences let us choose the 'initial' root λ_0^k. All the roots are equivalent in series A and D, but in series B one of the long

roots must be chosen (this will be justified later). For definiteness we fix the following choice (all the roots are written in the e-basis):

$$\lambda_0^k = \begin{cases} e_1 - e_2 & \text{for} \quad sl(N-2k) \\ e_1 + e_2 & \text{for} \quad so(2(N-2k)) \\ e_1 + e_2 & \text{for} \quad so(2(N-2k)+1). \end{cases} \tag{5.8}$$

Consider the set of roots orthogonal to λ_0^k. They form the subsystems for the following subalgebras in $\mathcal{A}_k$:

$$sl(M-2) \subset sl(M) \tag{5.9}$$

$$so(2M-4) \oplus sl(2) \subset so(2M) \tag{5.10}$$

$$so(2M-3) \oplus sl(2) \subset so(2M+1). \tag{5.11}$$

Notice that in all the cases (5.9)–(5.11) the corresponding universal enveloping algebras contain $\mathcal{A}_{k+1}$.

For each $\mathcal{A}_k$ let us form the set π_k of roots λ that are constituent for the initial root λ_0^k, i.e.

$$\pi_k = \left\{\lambda', \lambda'' \mid \lambda' + \lambda'' = \lambda_0^k;\ \lambda' + \lambda_0^k, \lambda' + \lambda_0^k \notin \Lambda_{\mathcal{A}}\right\}. \tag{5.12}$$

For each element $\lambda' \in \pi_k$ one can indicate the root $\lambda'' \in \pi_k$ such that $\lambda' + \lambda'' = \lambda_0^k$. Let us consider the ordered pairs (λ', λ'') and decompose π_k according to its main property

$$\begin{aligned} &\pi_k = \pi_k' \cup \pi_k'' \\ &\pi_k' = \{\lambda'\} \qquad \pi_k'' = \{\lambda''\}. \end{aligned} \tag{5.13}$$

For the sequences we are dealing with these sets are

$$\{\lambda_l', \lambda_l''\} = \begin{cases} \{\{(e_1 - e_l)\}, \{(e_l - e_2)\}\}_{l=3,4,\dots,M} & \text{for} \quad sl(M) \\ \{\{(e_1 \pm e_l)\}, \{(e_2 \pm e_l)\}\}_{l=3,4,\dots,M} & \text{for} \quad so(2M) \\ \{\{e_1, (e_1 \pm e_l)\}, \{e_2, (e_2 \pm e_l)\}\}_{l=3,4,\dots,M} & \text{for} \quad so(2M+1). \end{cases} \tag{5.14}$$

The important observation is that the generators $L_{\lambda'}$ and $L_{\lambda''}$ for $\lambda' \in \pi_k'$ and $\lambda'' \in \pi_k''$ form the bases for the spaces of conjugated defining representations of the subalgebras $\mathcal{A}_{k+1} \subset \mathcal{A}_k$ (with respect to the adjoint action). These subrepresentations are

$$\left\{\underline{(M-2)}, \underline{(M-2)}^*\right\} \qquad \text{for} \quad U(sl(M-2)) \subset U(sl(M)) \tag{5.15}$$

$$\left\{\underline{(2(M-2))}, \underline{(2(M-2))}\right\} \qquad \text{for} \quad U(so(2(M-2))) \subset U(so(2M)) \tag{5.16}$$

$$\left\{\underline{(2M-3)}, \underline{(2M-3)}\right\} \qquad \text{for} \quad U(so(2M-3)) \subset U(so(2M+1)). \tag{5.17}$$

Notice that any generator L_λ ($\lambda \in \pi_k$) commutes with $L_{\lambda_0^k}$ and with all the other elements $\{L_\mu | \mu \in \pi_k\}$ except its counterpart—the generator $L_{\lambda_0^k - \lambda}$. Together with L_λ we shall consider the Cartan generator $H_{\lambda_0^k}$ dual to the initial root (its functional $(H_{\lambda_0^k})^*$ is proportional to λ_0^k). To simplify the expressions we shall use the fact that in any classical Lie algebra there exists a basis where the structure constants for the generators $\{L_\lambda, H_{\lambda_0^k} | \lambda \in \pi_k\}$ can be normalized to form the following compositions:

$$\begin{aligned} &\left[H_{\lambda_0^k}, L_{\lambda'}\right] = \tfrac{1}{2} L_{\lambda'} && \left[L_{\lambda_0^k}, L_{\lambda'}\right] = \left[L_{\lambda_0^k}, L_{\lambda_0^k - \lambda'}\right] = 0 \\ &\left[H_{\lambda_0^k}, L_{\lambda_0^k - \lambda'}\right] = \tfrac{1}{2} L_{\lambda_0^k - \lambda'} && \left[H_{\lambda_0^k}, L_{\lambda_0^k}\right] = L_{\lambda_0^k} \\ &\left[L_{\lambda'}, L_{\lambda_0^k - \lambda'}\right] = L_{\lambda_0^k} && \lambda' \in \pi_k' \qquad \lambda_0^k - \lambda' \in \pi_k''. \end{aligned} \tag{5.18}$$

In the example considered in section 7 we present the explicit realizations for the generators of $\mathcal{A}_k$ that fit the relations above.

The relations (5.18) show that for every triple of roots $\{\lambda', \lambda_0^k - \lambda', \lambda_0^k\}$ we have the subalgebra $\boldsymbol{L}^{\lambda'}(\alpha, \beta)$ with $\alpha = \beta = \frac{1}{2}$ (see (3.2)). The set of generators

$$\{L_{\lambda|_{\lambda\in\pi_k}}, L_{\lambda_0^k}, H_{\lambda_0^k}\} \tag{5.19}$$

define a subalgebra $\mathcal{B}_k \subset \mathcal{A}_k$.

Let us perform in $\mathcal{A}_k$ the Jordanian twist

$$\Phi_{\mathcal{J}_k} = \exp\{H_{\lambda_0^k} \otimes \sigma_0^k\} \tag{5.20}$$

with $\sigma_0^k = \ln(1 + L_{\lambda_0^k})$. In the twisted algebra $(\mathcal{A}_k)_{\Phi_{\mathcal{J}_k}}$ the subalgebras $\{\boldsymbol{L}^{\lambda'}(\frac{1}{2}, \frac{1}{2}) \mid \lambda' \in \pi_k'\}$ described above obviously obey the conditions of proposition 1 and the corresponding sequence of twists

$$\Phi_{\mathcal{E}_k} = \prod_{\lambda'\in\pi_k'} \Phi_{\mathcal{E}\lambda'} = \prod_{\lambda'\in\pi_k'} \exp\{L_{\lambda'} \otimes L_{\lambda_0^k-\lambda'} \mathrm{e}^{-\frac{1}{2}\sigma_0^k}\} \tag{5.21}$$

can be performed in it. This gives for each $\mathcal{A}_k$ the following $\mathcal{ET}$ element:

$$\mathcal{F}_{\mathcal{B}_k} = \Phi_{\mathcal{E}_k}\Phi_{\mathcal{J}_k} = \Big(\prod_{\lambda'\in\pi_k'} \Phi_{\mathcal{E}\lambda'}\Big)\Phi_{\mathcal{J}_k}. \tag{5.22}$$

The sets of algebras $\mathcal{A}_k$ presented in (5.5)–(5.7) together with their subalgebras $\mathcal{B}_k$ (defined by the basic families (5.19)) form the correlated sequences of subalgebras that satisfy the conditions of proposition 2. To prove this let us consider the adjoint representation $\mathrm{ad}(\mathcal{A}_0) \equiv d_0$ and its restrictions to the subalgebras $\mathcal{A}_k$: $d_k = \mathrm{ad}(\mathcal{A}_0)_{|\mathcal{A}_k}$. The space of $\mathcal{B}_k$ is invariant with respect to d_{k+1}. It contains the subspaces of two trivial subrepresentations (generated by $L_{\lambda_0^k}$ and $H_{\lambda_0^k}$). This means that the $\mathcal{JT}$ factor $\Phi_{\mathcal{J}_k}$ commutes with the algebra $\Delta(\mathcal{A}_{k+1}) \subset \mathcal{A}_{k+1} \otimes \mathcal{A}_{k+1}$. The other two invariant subspaces in $\mathcal{B}_k$ refer to the fundamental representations of $\mathcal{A}_{k+1}$ indicated in (5.15)–(5.17). Due to the commutation relations in $\mathcal{B}_k$ the element $\ln \Phi_{\mathcal{E}_k}$ can be written as

$$\Big(\sum_{\lambda'\in\pi_k'} L_{\lambda'} \otimes L_{\lambda_0^k-\lambda'}\Big)\mathrm{e}^{-\frac{1}{2}\sigma_0^k}. \tag{5.23}$$

With the ordered pairs of roots as in (5.14) this expression is d_{k+1}-invariant (the converted bases for conjugated representations modulo the scalar factor). We have arrived at the conclusion that the sets of subalgebras $\mathcal{A}_k$ (equations (5.5)–(5.7)) and $\mathcal{B}_k$ (defined by (5.12), (5.14) and (5.19)) with the twisting elements $\mathcal{F}_{\mathcal{B}_k}$ (5.22) satisfy the conditions of proposition 2. Thus for any classical simple Lie algebra of the series A, B and D the chains of twists $\mathcal{F}_{\mathcal{B}_{k<0}} \equiv \mathcal{F}_{\mathcal{B}_k}\mathcal{F}_{\mathcal{B}_{k-1}} \dots \mathcal{F}_{\mathcal{B}_0}$ $(k = 0, 1, \dots, p)$ exist.

The twisting element for a chain can be written explicitly as

$$\begin{aligned}\mathcal{F}_{\mathcal{B}_{k<0}} = {} & \prod_{\lambda'\in\pi_k'} \big(\exp\{L_{\lambda'} \otimes L_{\lambda_0^k-\lambda'}\mathrm{e}^{-\frac{1}{2}\sigma_0^k}\}\big) \exp\{H_{\lambda_0^k} \otimes \sigma_0^k\} \\ & \times \prod_{\lambda'\in\pi_{k-1}'} \big(\exp\{L_{\lambda'} \otimes L_{\lambda_0^{k-1}-\lambda'}\mathrm{e}^{-\frac{1}{2}\sigma_0^{k-1}}\}\big) \exp\{H_{\lambda_0^{k-1}} \otimes \sigma_0^{k-1}\} \\ & \vdots \\ & \times \prod_{\lambda'\in\pi_0'} \big(\exp\{L_{\lambda'} \otimes L_{\lambda_0^0-\lambda'}\mathrm{e}^{-\frac{1}{2}\sigma_0^0}\}\big) \exp\{H_{\lambda_0^0} \otimes \sigma_0^0\}.\end{aligned} \tag{5.24}$$

Any number of exponential factors can be cut out from the left. The remaining part always conserves the property of the twisting element for the corresponding classical Lie algebra. When on the left-hand side one has a product of extensions that is not complete (not all $\lambda' \in \pi'_k$ are used),

$$\mathcal{F}_{\mathcal{E}_\Theta}\mathcal{F}_{\mathcal{B}_{(k-1)\prec 0}} = \prod_{\lambda'\in\Theta\subset\pi'_k}\left(\exp\left\{L_{\lambda'}\otimes L_{\lambda_0^k-\lambda'}\mathrm{e}^{-\frac{1}{2}\sigma_0^k}\right\}\right)\exp\{H_{\lambda_0^k}\otimes\sigma_0^k\}\mathcal{F}_{\mathcal{B}_{k-1}}\cdots\mathcal{F}_{\mathcal{B}_0} \tag{5.25}$$

the subalgebra $\mathcal{A}_{k+1}$ will be twisted non-trivially by such an element. In this case the twisting deformation with $\mathcal{F}_{\mathcal{B}_{k+1}}$ (of the (5.22) type) cannot be applied to $\mathcal{A}_{\mathcal{F}_{\mathcal{E}_\Theta}\mathcal{F}_{\mathcal{B}_{k\prec 0}}}$. The necessary *primitivization of generators in $\mathcal{A}_{k+1}$ is regained when the product of extensions is complete* and forms an invariant of the representation d_{k+1}. We call this the 'matreshka' effect.

Quantizations $\mathcal{A}_{\mathcal{F}_{\mathcal{B}_{p\prec 0}}}$ of classical Lie algebras produce the chains of $\mathcal{R}$-matrices:

$$\mathcal{R}_{\mathcal{B}_{p\prec 0}} = (\mathcal{F}_{\mathcal{B}_p})_{21}(\mathcal{F}_{\mathcal{B}_{p-1}})_{21}\ldots(\mathcal{F}_{\mathcal{B}_0})_{21}\mathcal{F}_{\mathcal{B}_0}^{-1}\ldots\mathcal{F}_{\mathcal{B}_{p-1}}^{-1}\mathcal{F}_{\mathcal{B}_p}^{-1}. \tag{5.26}$$

The explicit expressions in terms of generators can be obtained by substituting the elements L_λ and $H_{\lambda_0^k}$ in (5.24) by the corresponding generators according to the prescription of roots in (5.8) and (5.14).

If the deformation parameter is introduced (as in (1.2)) the chains of classical r-matrices can be extracted from (5.26):

$$r_{\mathcal{B}_{p\prec 0}} = \sum_{k=0,1,\ldots,p}\left(H_{\lambda_0^k}\wedge L_{\lambda_0^k} + \sum_{\lambda'\in\pi_k} L_{\lambda'}\wedge L_{\lambda_0^k-\lambda'}\right). \tag{5.27}$$

With the obvious modifications of factors (summands) the sequences of $\mathcal{R}$-matrices (classical r-matrices) for incomplete chains of twists can also be written.

6. Improper chains. Symplectic algebras

Imposing additional conditions on the internal structure of the Hopf algebras involved one can minimize the algebra $\mathcal{A}_0$ on which the chain is based to the universal enveloping algebra $\mathcal{A}_0^{\mathrm{car}} \equiv U(\mathfrak{g}_0^{\mathrm{car}})$ of the carrier of the chain. This happens, for example, when $\mathfrak{g}_0$ in $\mathcal{A}_0 = U(\mathfrak{g}_0)$ is a sequence of semidirect sums and every $\mathcal{B}_k$ is a $\mathcal{B}_{k+1}$-module with respect to the adjoint action (in $\mathfrak{g}_0$),

$$\begin{aligned}&\mathfrak{g}_0^{\mathrm{car}} = \mathfrak{g}_0 = \mathcal{B}_p \vdash (\mathcal{B}_{p-1}\vdash(\cdots\vdash\mathcal{B}_0)\cdots)\\ &\left[\mathcal{B}_{k+1},\mathcal{B}_k\right]\subset\mathcal{B}_k.\end{aligned} \tag{6.1}$$

In this case one can define the subalgebras $\mathcal{A}_k^{\mathrm{car}}$ as

$$U(\mathfrak{g}_k^{\mathrm{car}}) = U(\mathcal{B}_p\vdash(\mathcal{B}_{p-1}\vdash(\cdots\vdash\mathcal{B}_k)\cdots)). \tag{6.2}$$

In the sequences of classical algebras that we considered in (5.5)–(5.7) the conditions (6.1) are fulfilled (with $\mathcal{B}_k$ defined by (5.19)). One can rewrite the sequences (5.1) for the classical Lie algebras so that the elements $\mathcal{A}_k$ will be substituted by $\mathcal{A}_k^{\mathrm{car}} \equiv U(\mathfrak{g}_k^{\mathrm{car}})$ defined by (6.2) and $\mathfrak{g}_0^{\mathrm{car}}$ will be the carrier of $\mathcal{F}_{\mathcal{B}_{p\prec 0}}$. There rests some freedom in choosing the initial root. Using it one can, in particular, place the carrier of the chain in the Borel subalgebra of the corresponding classical Lie algebra. For example, in the case of $sl(N)$ the carrier subalgebra of the full chain of the type (5.24) can be arranged to contain all the generators with the positive root vectors and a part of the Cartan subalgebra (spanned by $H_{1,N}$, $H_{2,N-1}$, ...).

For simple Lie algebras the chain carrier subalgebra covers only a proper subspace of an algebra. The chains $\mathcal{F}_{\mathcal{B}_{p\prec 0}}$ that we described in the previous section are maximal in the sense

that $\mathcal{A}_0^{\rm car}$ is a maximal Frobenius subalgebra in the corresponding classical Lie algebra that can be composed from the subalgebras of the type (5.18) (that is, using $\Phi_{\mathcal{E}_k}$ and $\Phi_{\mathcal{J}_k}$ as elementary blocks). These chains are also specific for the simple algebras we are dealing with. In each of the three cases (equations (5.5)–(5.7)) the individual properties of the root system are used to form a chain.

The universal enveloping algebras for other simple Lie algebras (the series C_N and the exceptional algebras) do not refer to the class of algebras conserving symmetric forms (over a field) and cannot be supplied by a specific chain of extended twists. Nevertheless, the quantization by a chain can be performed in these algebras using the classical subalgebras of the series A, B and D contained in them. For example, due to the inclusion $sl(N) \subset sp(N)$ the chain specific to $U(sl(N))$ can be used to quantize $U(sp(N))$. Such chains can be called improper.

Now we shall study the universal enveloping algebras for symplectic simple Lie algebras (the series C_N) where the maximal chain appears to be improper. (It exploits almost exclusively the A_{N-1} subalgebra in C_N.) In the e-basis the $sp(N)$ roots can be fixed as follows:

$$\Lambda_{sp(N)} = \begin{cases} e_i - e_j & \text{for} \quad i \neq j \\ \pm(e_i + e_j) & \text{for} \quad i \leqslant j \end{cases} \qquad i, j = 1, 2, \ldots, N. \tag{6.3}$$

Whatever root will be chosen as the initial one the extensions will contain generators whose roots will have the non-zero projections on the root system of $sp(N-2)$.

Note that if we fix a short root to be the initial one, for example $\lambda_0^0 = e_i - e_j$, there will be pairs of constituent roots that do not satisfy the conditions (5.12). The generators corresponding to $\lambda' = -2e_j$, $\lambda'' = e_i + e_j$ and λ_0^0 do not form a subalgebra of $\boldsymbol{L}(\alpha, \beta)$ type. Thus we are to consider the subalgebra $\mathcal{A}_1 = U(sp(N-2))$. The generators corresponding to π_k' and π_k'' (5.12),

$$\{e_i \pm e_l\} \qquad \text{and} \qquad \{-e_j \pm e_l\} \qquad l = 3, \ldots, N$$

form the bases for the defining representations of $sp(N-2)$. The symplectic invariant

$$\sum_l (L_{e_i+e_l} \otimes L_{-e_j-e_l} - L_{e_i-e_l} \otimes L_{-e_j+e_l}) \tag{6.4}$$

does not correlate with the $\mathcal{ET}$ (3.5). Otherwise one can check that the extensions based on linear combinations of the type (6.4) (with the coefficients in $\boldsymbol{C}[[\sigma]]$) violate the twist equation (2.3).

We can diminish the subalgebra $\mathcal{A}_1$ and put $\mathcal{A}_1 = U(sl(N-2))$. In this case the summands $\sum_l L_{e_i+e_l} \otimes L_{-e_j-e_l}$ and $\sum_l L_{e_i-e_l} \otimes L_{-e_j+e_l}$ in (6.4) will be separately invariant with respect to $\mathcal{A}_1$ and both will match with the sequences of extensions (3.5). In such a way we can proceed by constructing the chain of extended twists for $U(sp(N))$ but this will be specific for A_n rather than for the C_n root system (except that the long root can be chosen to be the first initial root).

7. Example. Maximal chain for $U(so(9))$

To illustrate the properties of chains we apply the algorithm presented in sections 5 and 6 to construct a chain of $\mathcal{ET}$'s for the algebra $U(so(9))$.

In this case the sequence (5.7) consists of two elements:

$$\mathcal{A}_1 \subset \mathcal{A}_0 = so(9) \supset so(5) \tag{7.1}$$

with the initial roots

$$\lambda_0^0 = e_1 + e_2 \qquad \lambda_0^1 = e_3 + e_4$$

and the corresponding sets of constituent roots

$$\pi_0' = \{\lambda^{0'}\} = \{e_1, e_1 \pm e_3, e_1 \pm e_4\}$$
$$\pi_0'' = \{\lambda^{0''}\} = \{e_2, e_2 \pm e_3, e_2 \pm e_4\}$$
$$\pi_1' = \{\lambda^{1'}\} = \{e_3\}$$
$$\pi_1'' = \{\lambda^{1''}\} = \{e_4\}.$$

The roots π_0' and π_0'' form the weight diagrams for the vector representations of $\mathcal{A}_1 = so(5)$.

Together with the Cartan generators $H_{\lambda_0^0}$, $H_{\lambda_0^1}$ the basic elements $E_{\lambda_0^0}$, $E_{\lambda_0^1}$ and $\{E_\lambda | \lambda \in \pi_0 \cup \pi_1\}$ form the 16-dimensional subalgebra $\mathfrak{g}_0^{\text{car}} \subset \mathfrak{g}_0 = so(9)$. It has the structure of a semidirect sum $\mathfrak{g}_0^{\text{car}} \approx \mathcal{B}_1 \vdash \mathcal{B}_0$. This means that studying this chain we can restrict ourselves to the subalgebra $U(\mathfrak{g}_0^{\text{car}})$.

The maximal chain for the sequence (7.1) has the following structure:

$$\mathcal{F}_{\mathcal{B}_{1\prec 0}} = \Phi_{\mathcal{E}_1}\Phi_{\mathcal{J}_1}\Phi_{\mathcal{E}_0}\Phi_{\mathcal{J}_0} = \Phi_{\mathcal{E}e_3}\Phi_{\mathcal{J}_1}\left(\prod_{\lambda' \in \pi_0'} \Phi_{\mathcal{E}\lambda'}\right)\Phi_{\mathcal{J}_0}. \tag{7.2}$$

The generators of $\mathfrak{g}_0$ can be expressed in terms of the antisymmetric Okubo matrices M_{ik}:

$$\boldsymbol{L}^{12} = \begin{cases} H_{12} = (-\mathrm{i}/2)(M_{12} + M_{34}) \\ E_1 = M_{29} - \mathrm{i}M_{19} \qquad E_2 = M_{49} - \mathrm{i}M_{39} \\ E_{1+2} = -M_{24} + \mathrm{i}M_{23} + \mathrm{i}M_{14} + M_{13}, \end{cases}$$

$$\begin{aligned} E_{1+3} &= -M_{26} + \mathrm{i}M_{25} + \mathrm{i}M_{16} + M_{15} \\ E_{1+4} &= -M_{28} + \mathrm{i}M_{27} + \mathrm{i}M_{18} + M_{17} \\ E_{2+3} &= -M_{46} + \mathrm{i}M_{45} + \mathrm{i}M_{36} + M_{35} \\ E_{2+4} &= -M_{48} + \mathrm{i}M_{47} + \mathrm{i}M_{38} + M_{37} \\ E_{1-3} &= -M_{26} - \mathrm{i}M_{25} + \mathrm{i}M_{16} - M_{15} \\ E_{1-4} &= -M_{28} - \mathrm{i}M_{27} + \mathrm{i}M_{18} - M_{17} \\ E_{2-3} &= -M_{46} - \mathrm{i}M_{45} + \mathrm{i}M_{36} - M_{35} \\ E_{2-4} &= -M_{48} - \mathrm{i}M_{47} + \mathrm{i}M_{38} - M_{37} \end{aligned} \tag{7.3}$$

$$\boldsymbol{L}^{34} = \begin{cases} H_{34} = (-\mathrm{i}/2)(M_{56} + M_{78}), \\ E_3 = M_{69} - \mathrm{i}M_{59} \qquad E_4 = M_{89} - \mathrm{i}M_{79} \\ E_{3+4} = -M_{68} + \mathrm{i}M_{67} + \mathrm{i}M_{58} + M_{57}. \end{cases}$$

Here the lower indices of the generators E indicate the corresponding $so(9)$ roots. The set of generators $\boldsymbol{L}^{12}$ ($\boldsymbol{L}^{34}$) forms the four-dimensional subalgebra of the type $\boldsymbol{L}(\frac{1}{2}, \frac{1}{2})$ with $E = E_{1+2}$ ($E = E_{3+4}$).

The explicit expressions for the main factors of the chain in this basis are as follows:

$$\begin{aligned}
&\Phi_{\mathcal{J}_0} = \exp(H_{12} \otimes \sigma_{12}) \qquad \Phi_{\mathcal{J}_1} = \exp(H_{34} \otimes \sigma_{34}) \\
&\Phi_{\mathcal{E}_0} = \exp\big((E_1 \otimes E_2 + \tfrac{1}{2}(E_{1-3} \otimes E_{2+3} + E_{1+3} \otimes E_{2-3} \\
&\qquad\qquad + E_{1-4} \otimes E_{2+4} + E_{1+4} \otimes E_{2-4}))(1 \otimes \mathrm{e}^{-\frac{1}{2}\sigma_{12}})\big) \\
&\Phi_{\mathcal{E}_1} = \exp(E_3 \otimes E_4 \mathrm{e}^{-\frac{1}{2}\sigma_{34}})
\end{aligned} \tag{7.4}$$

with

$$\begin{aligned}
\sigma_{12} &= \sigma_0^0 = \ln(1 + E_{1+2}) \\
\sigma_{34} &= \sigma_0^1 = \ln(1 + E_{3+4}).
\end{aligned}$$

After the first Jordanian twisting,

$$\mathcal{A}_0^{\mathrm{car}} \xrightarrow{\Phi_{\mathcal{J}_0}} (\mathcal{A}_0^{\mathrm{car}})_{J_0}$$

the subalgebra

$$\boldsymbol{L}^{34} = \mathcal{B}_1 \subset \mathcal{A}$$

remains primitive. The carrier subalgebra for $\Phi_{\mathcal{J}_0}$ acquires the coproducts

$$\begin{aligned}
\Delta_{J_0}(H_{12}) &= H_{12} \otimes \mathrm{e}^{-\sigma_{12}} + 1 \otimes H_{12} \\
\Delta_{J_0}(E_{1+2}) &= E_{1+2} \otimes \mathrm{e}^{\sigma_{12}} + 1 \otimes E_{1+2}.
\end{aligned} \tag{7.5}$$

The coproducts for the remaining generators of $\mathcal{B}_0$ are of the form

$$\Delta_{J_0}(E) = E \otimes \mathrm{e}^{\frac{1}{2}\sigma_{12}} + 1 \otimes E. \tag{7.6}$$

Among the exponential factors $\Phi_{\mathcal{E}\lambda'}$ of the extension $\Phi_{\mathcal{E}_0}$ (see (7.2)) there is one ($\Phi_{\mathcal{E}e_1}$) that does not touch the subalgebra $\boldsymbol{L}^{34}$. Each of the rest $\{\Phi_{\mathcal{E}\lambda'|\lambda'=e_1\pm e_3,e_1\pm e_4}\}$ being applied separately produces a non-trivial deformation of $\boldsymbol{L}^{34}$. These extensions can be combined to form the $so(5)$-invariant (see (7.4)). In this case, i.e. after the twisting

$$(\mathcal{A}_0^{\mathrm{car}})_{J_0} \xrightarrow{\Phi_{\mathcal{E}_0}} (\mathcal{A}_0^{\mathrm{car}})_{\mathcal{E}_0 J_0}$$

the primitivity of generators in $\boldsymbol{L}^{34}$ is restored. The coproducts for the generators of $(\mathcal{B}_0)_{\mathcal{E}_0 J_0}$ are deformed according to the general rule (see section 2),

$$\begin{aligned}
&\Delta_{\mathcal{E}_0 J_0}(E_{\lambda'}) = E_{\lambda'} \otimes \mathrm{e}^{-\frac{1}{2}\sigma_{12}} + 1 \otimes E_{\lambda'} \\
&\Delta_{\mathcal{E}_0 J_0}(E_{\lambda_0^0 - \lambda'}) = E_{\lambda_0^0 - \lambda'} \otimes \mathrm{e}^{\frac{1}{2}\sigma_{12}} + \mathrm{e}^{\sigma_{12}} \otimes E_{\lambda_0^0 - \lambda'} \\
&\Delta_{\mathcal{E}_0 J_0}(E_{\lambda_0^0}) = E_{\lambda_0^0} \otimes \mathrm{e}^{\sigma_{12}} + 1 \otimes E_{\lambda_0^0} \\
&\Delta_{\mathcal{E}_0 J_0}(H_{12}) = H_{12} \otimes \mathrm{e}^{-\sigma_{12}} + 1 \otimes H_{12} - E_1 \otimes E_2 \mathrm{e}^{-\frac{3}{2}\sigma_{12}} \\
&\qquad\qquad - \tfrac{1}{2} \sum_{\lambda'=e_1\pm e_3, e_1\pm e_4} E_{\lambda'} \otimes E_{\lambda_0^0 - \lambda'} \mathrm{e}^{-\frac{3}{2}\sigma_{12}}.
\end{aligned} \tag{7.7}$$

Due to the 'matreshka' effect, the second Jordanian twist can be applied to $(\mathcal{A}_0^{\mathrm{car}})_{\mathcal{E}_0 J_0}$

$$(\mathcal{A}_0^{\mathrm{car}})_{\mathcal{E}_0 J_0} \xrightarrow{\Phi_{\mathcal{J}_1}} (\mathcal{A}_0^{\mathrm{car}})_{J_1 \mathcal{E}_0 J_0}.$$

This leads to the following deformations:

- the subalgebra $\mathcal{B}_1$ acquires the well known twisted form with the defining coproducts

$$\begin{aligned}
&\Delta_{J_1\mathcal{E}_0J_0}(H_{34}) = H_{34}\otimes \mathrm{e}^{-\sigma_{34}} + 1\otimes H_{34}\\
&\Delta_{J_1\mathcal{E}_0J_0}(E_{3+4}) = E_{3+4}\otimes \mathrm{e}^{\sigma_{34}} + 1\otimes E_{3+4}\\
&\Delta_{J_1\mathcal{E}_0J_0}(E_k) = E_k\otimes \mathrm{e}^{\frac{1}{2}\sigma_{34}} + 1\otimes E_k \qquad k=3,4
\end{aligned} \tag{7.8}$$

- the subalgebra $(\boldsymbol{L}^{12})_{\mathcal{E}_0J_0}$ rests untouched,

$$(\boldsymbol{L}^{12})_{\mathcal{E}_0J_0} = (\boldsymbol{L}^{12})_{J_1\mathcal{E}_0J_0}$$

- for each $\{\lambda = e_i \pm e_k \mid i=1,2;\ k=3,4\}$ the following substitution is performed in the coproducts for the generators E_λ:

$$E_\lambda\otimes f(\sigma_{12}) \quad\longrightarrow\quad E_\lambda\otimes \mathrm{e}^{\pm\frac{1}{2}\sigma_{34}} f(\sigma_{12})$$

- in $\Delta_{J_1\mathcal{E}_0J_0}(E_\lambda)$ for each $\{\lambda = e_i - e_k \mid i=1,2;\ k=3,4\}$ the additional term appears,

$$(-1)^{k+1} H_{34}\mathrm{e}^{(i-1)\sigma_{12}}\otimes E_{e_i+e_{\overline{k}}}\mathrm{e}^{-\sigma_{34}}$$

(here $\overline{3}=4, \overline{4}=3$);

- for the Cartan generator H_{12} the coproduct becomes

$$\begin{aligned}
\Delta_{J_1\mathcal{E}_0J_0}(H_{12}) = {}& H_{12}\otimes \mathrm{e}^{-\sigma_{12}} + 1\otimes H_{12} - E_1\otimes E_2\mathrm{e}^{-\frac{3}{2}\sigma_{12}}\\
&-\tfrac{1}{2}E_{1+3}\otimes E_{2-3}\mathrm{e}^{\frac{1}{2}\sigma_{34}-\frac{3}{2}\sigma_{12}} - H_{34}E_{1+3}\otimes E_{2+4}\mathrm{e}^{-\frac{1}{2}\sigma_{34}-\frac{3}{2}\sigma_{12}}\\
&-\tfrac{1}{2}E_{1+4}\otimes E_{2-4}\mathrm{e}^{\frac{1}{2}\sigma_{34}-\frac{3}{2}\sigma_{12}} + H_{34}E_{1+4}\otimes E_{2+3}\mathrm{e}^{-\frac{1}{2}\sigma_{34}-\frac{3}{2}\sigma_{12}}\\
&-\tfrac{1}{2}E_{1-4}\otimes E_{2+4}\mathrm{e}^{-\frac{1}{2}\sigma_{34}-\frac{3}{2}\sigma_{12}} - \tfrac{1}{2}E_{1-3}\otimes E_{2+3}\mathrm{e}^{-\frac{1}{2}\sigma_{34}-\frac{3}{2}\sigma_{12}}.
\end{aligned}$$

The last twisting (that completes the chain $\mathcal{B}_{1\prec 0}$),

$$(\mathcal{A}_0^{\mathrm{car}})_{J_1\mathcal{E}_0J_0} \xrightarrow{\Phi_{\mathcal{E}_1}} (\mathcal{A}_0^{\mathrm{car}})_{\mathcal{B}_{1\prec 0}} \tag{7.9}$$

does not change the coproducts for the generators $\{E_{i+k} \mid i=1,2;\ k=3,4\}$. It produces the ordinary transformation for $\boldsymbol{L}^{34}$,

$$\begin{aligned}
&\Delta_{\mathcal{B}_{1\prec 0}}(H_{34}) = \Delta_{J_1\mathcal{E}_0J_0}(H_{34}) + E_3\otimes E_4\mathrm{e}^{-\frac{3}{2}\sigma_{34}}\\
&\Delta_{\mathcal{B}_{1\prec 0}}(E_3) = E_3\otimes \mathrm{e}^{-\frac{1}{2}\sigma_{34}} + 1\otimes E_3\\
&\Delta_{\mathcal{B}_{1\prec 0}}(E_4) = E_4\otimes \mathrm{e}^{\frac{1}{2}\sigma_{34}} + \mathrm{e}^{\sigma_{34}}\otimes E_4.
\end{aligned}$$

The generators E_1, E_2, E_{i-k} and H_{12} are non-trivially twisted by the transformation (7.9),

$$\Delta_{\mathcal{B}_{1\prec 0}}(E_1) = \Delta_{J_1\mathcal{E}_0J_0}(E_1) - E_{1+3}\otimes E_4\mathrm{e}^{-\frac{1}{2}\sigma_{34}-\frac{1}{2}\sigma_{12}} - E_3\otimes E_{1+4}\mathrm{e}^{-\frac{1}{2}\sigma_{34}}$$

$$\Delta_{\mathcal{B}_{1\prec 0}}(E_2) = \Delta_{J_1\mathcal{E}_0J_0}(E_2) - E_{2+3}\otimes E_4\mathrm{e}^{-\frac{1}{2}\sigma_{34}+\frac{1}{2}\sigma_{12}} - E_3\mathrm{e}^{\sigma_{12}}\otimes E_{2+4}\mathrm{e}^{-\frac{1}{2}\sigma_{34}}$$

$$\begin{aligned}
\Delta_{\mathcal{B}_{1\prec 0}}(E_{1-3}) = {}& \Delta_{J_1\mathcal{E}_0J_0}(E_{1-3}) + 2E_1\otimes E_4\mathrm{e}^{-\sigma_{34}-\frac{1}{2}\sigma_{12}}\\
&-E_{1+3}\otimes E_4^2\mathrm{e}^{-\frac{1}{2}\sigma_{12}-\frac{3}{2}\sigma_{34}} - 2E_3\otimes E_4E_{1+4}\mathrm{e}^{-\frac{3}{2}\sigma_{34}}
\end{aligned}$$

$$\begin{aligned}
\Delta_{\mathcal{B}_{1\prec 0}}(E_{2-3}) = {}& \Delta_{J_1\mathcal{E}_0J_0}(E_{2-3}) + 2E_2\otimes E_4\mathrm{e}^{-\sigma_{34}+\frac{1}{2}\sigma_{12}}\\
&-E_{2+3}\otimes E_4^2\mathrm{e}^{\frac{1}{2}\sigma_{12}-\frac{3}{2}\sigma_{34}} - 2E_3\mathrm{e}^{\sigma_{12}}\otimes E_4E_{2+4}\mathrm{e}^{-\frac{3}{2}\sigma_{34}}
\end{aligned}$$

$$\Delta_{\mathcal{B}_{1\prec 0}}(E_{1-4}) = \Delta_{J_1\mathcal{E}_0J_0}(E_{1-4}) + 2E_3\otimes E_1\mathrm{e}^{-\frac{1}{2}\sigma_{34}} - E_3^2\otimes E_{1+4}\mathrm{e}^{-\sigma_{34}} + 2E_3\otimes E_4E_{1+3}\mathrm{e}^{-\frac{3}{2}\sigma_{34}}$$

$$\begin{aligned}
\Delta_{\mathcal{B}_{1\prec 0}}(E_{2-4}) = {}& \Delta_{J_1\mathcal{E}_0J_0}(E_{2-4}) + 2E_3\mathrm{e}^{\sigma_{12}}\otimes E_2\mathrm{e}^{-\frac{1}{2}\sigma_{34}}\\
&-E_3^2\mathrm{e}^{\sigma_{12}}\otimes E_{2+4}\mathrm{e}^{-\sigma_{34}} + 2E_3\mathrm{e}^{\sigma_{12}}\otimes E_4E_{2+3}\mathrm{e}^{-\frac{3}{2}\sigma_{34}}
\end{aligned}$$

$$\Delta_{\mathcal{B}_{1<0}}(H_{12}) = \Delta_{J_1\mathcal{E}_0J_0}(H_{12}) + (E_{1+3}\otimes E_2E_4 + E_3E_1\otimes E_{2+4})1\otimes \mathrm{e}^{-\frac{1}{2}\sigma_{34}-\frac{3}{2}\sigma_{12}}$$
$$-(E_1+E_{1+4}E_3)\otimes E_{2+3}E_4\mathrm{e}^{-\sigma_{34}-\frac{3}{2}\sigma_{12}} + \tfrac{1}{2}E_{1+3}\otimes E_{2+3}(E_4)^2\mathrm{e}^{-\frac{3}{2}(\sigma_{34}+\sigma_{12})}$$
$$-E_{1+4}E_3\otimes E_2\mathrm{e}^{-\frac{3}{2}\sigma_{12}} + \tfrac{1}{2}E_{1+4}(E_3)^2\otimes E_{2+4}\mathrm{e}^{-\frac{1}{2}\sigma_{34}-\frac{3}{2}\sigma_{12}}.$$

These relations complete the description of the twisted Hopf algebra $(\mathcal{A}_0^{\mathrm{car}})_{\mathcal{B}_{1<0}}$. Using the explicit expressions (7.4) for the chain factors one can reconstruct the Hopf algebra $U(so(9))_{\mathcal{B}_{1<0}}$ containing $(\mathcal{A}_0^{\mathrm{car}})_{\mathcal{B}_{1<0}}$. Both of them are triangular with the universal element

$$\mathcal{R}_{\mathcal{B}_{k<0}} = (\Phi_{\mathcal{E}_1}\Phi_{\mathcal{J}_1}\Phi_{\mathcal{E}_0}\Phi_{\mathcal{J}_0})_{21}(\Phi_{\mathcal{E}_1}\Phi_{\mathcal{J}_1}\Phi_{\mathcal{E}_0}\Phi_{\mathcal{J}_0})^{-1}.$$

The deformation parameter can be introduced so that the classical r-matrix

$$r_{\mathcal{B}_{1<0}} = H_{12}\wedge E_{1+2} + H_{34}\wedge E_{3+4} + E_1\wedge E_2 + E_3\wedge E_4$$
$$+\tfrac{1}{2}(E_{1-3}\wedge E_{2+3} + E_{1+3}\wedge E_{2-3} + E_{1-4}\wedge E_{2+4} + E_{1+4}\wedge E_{2-4})$$

determines the Lie–Poisson structure that was quantized explicitly by the chain of twists $\mathcal{F}_{\mathcal{B}_{k<0}}$.

8. Conclusions

Chains of twists provide a rich variety of new quantizations for a certain class of Lie algebras described in proposition 2. As was demonstrated in [17] extended twists can be accompanied by the special Reshetikhin twists which 'rotate' the roots of the carrier subalgebras for the Jordanian factors. It is easy to check that such 'rotations' can also be applied in the case of chains. The corresponding additional factors $\mathcal{F}_{\mathcal{R}} = \exp\big(\big(H_{\lambda_0^{k-1}} + \theta(H_{\lambda_0^{k-1}})^{\perp}\big)\otimes\sigma_0^{k-1}\big)$ (here $(H_{\lambda_0^{k-1}})^{\perp}$ is orthogonal to $H_{\lambda_0^k}$ and $H_{\lambda_0^{k-1}}$) can be included in each $\mathcal{F}_{\mathcal{B}_{k-1}}$. It can be shown that though the factor $\Phi_{\mathcal{E}_{k-1}}$ must be changed its invariance properties with respect to $\mathcal{B}_k$ can be conserved. In this context the chains are flexible and their multiparametric versions can be easily constructed.

The deformation parameters can be introduced in chains by rescaling the generators of the subalgebra $\mathcal{B}_k$. It must be stressed that each $\mathcal{B}_k$ can be rescaled separately with an independent variable ξ_k. When all these rescaling factors are proportional to the deformation parameter ξ, i.e. $\xi_k = \xi\eta_k$, then in the classical limit the parameters η_k appear as the multipliers in the classical r-matrix (compare with (5.27)):

$$r_{\mathcal{B}_{p<0}} = \sum_{k=0,1,\ldots,p}\eta_k\left(H_{\lambda_0^k}\wedge L_{\lambda_0^k} + \sum_{\lambda'\in\pi_k}L_{\lambda'}\wedge L_{\lambda_0^k-\lambda'}\right).$$

The mechanisms described above can be combined together both leading to the multiparametric versions of chains.

One of the consequences of proposition 2 is that for a large set of universal enveloping algebras (including A, B and D series of classical algebras), classical r-matrices of the type (5.27) exist. For the special case of $\mathfrak{g} = sl(N)$ they were first presented in [8]. As we have shown above they originate from the specific properties of extended Jordanian twists, the possibility to form chains for certain types of universal enveloping algebras.

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